Robust Practical Nonlinear Control for a Multi-stage Inter-cooled Compression System

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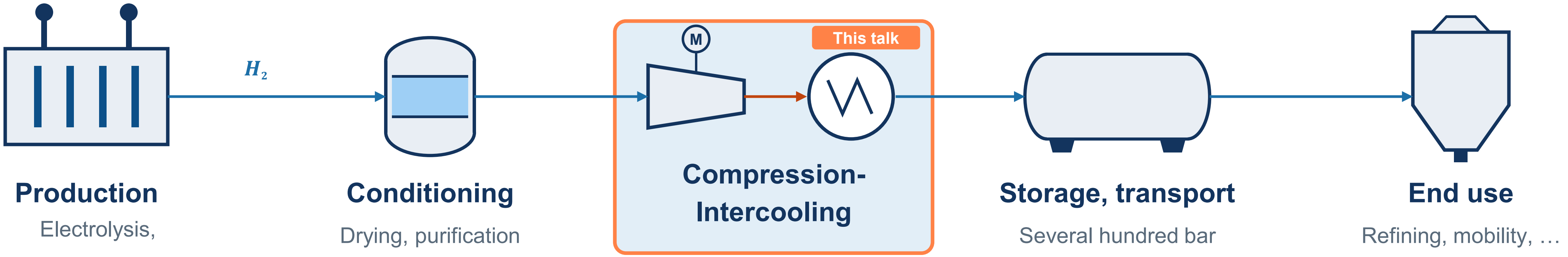
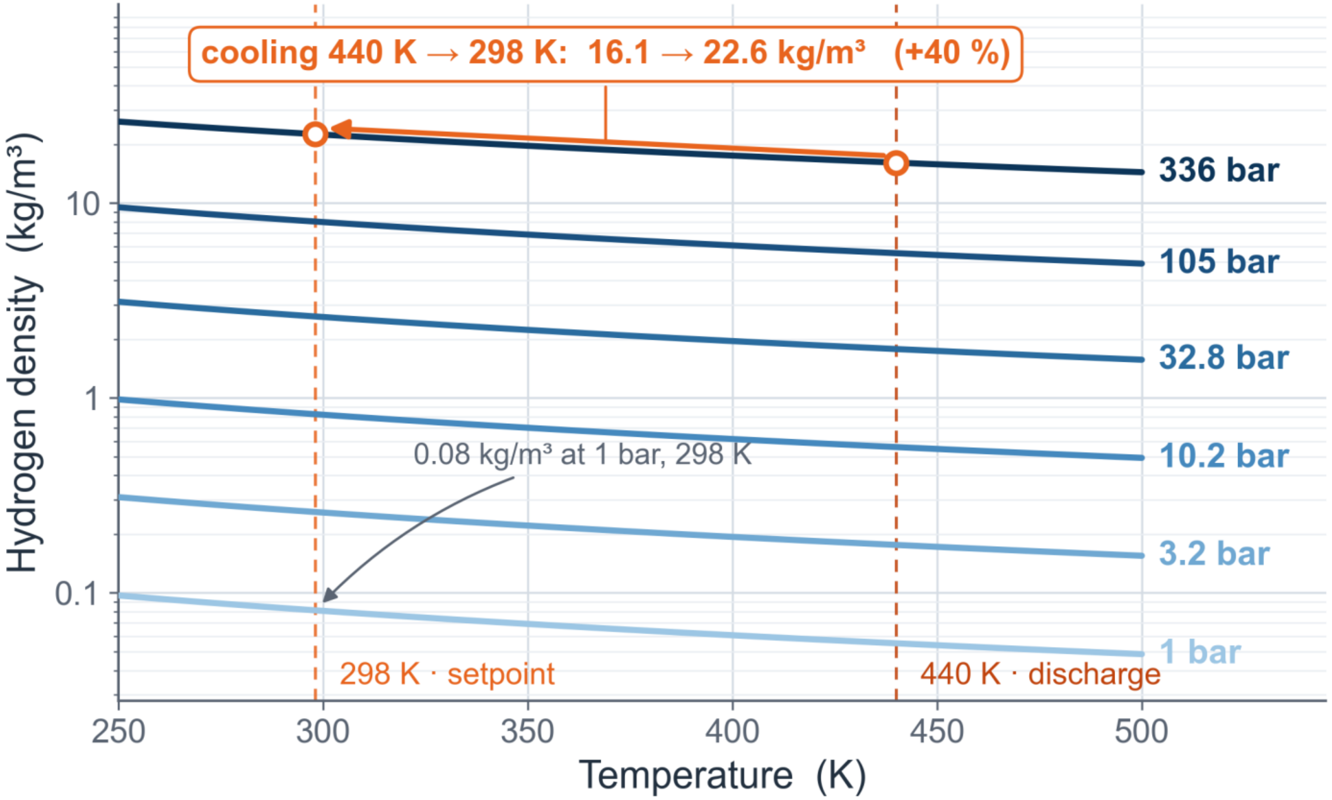
Hydrogen only becomes storable when it is compressed

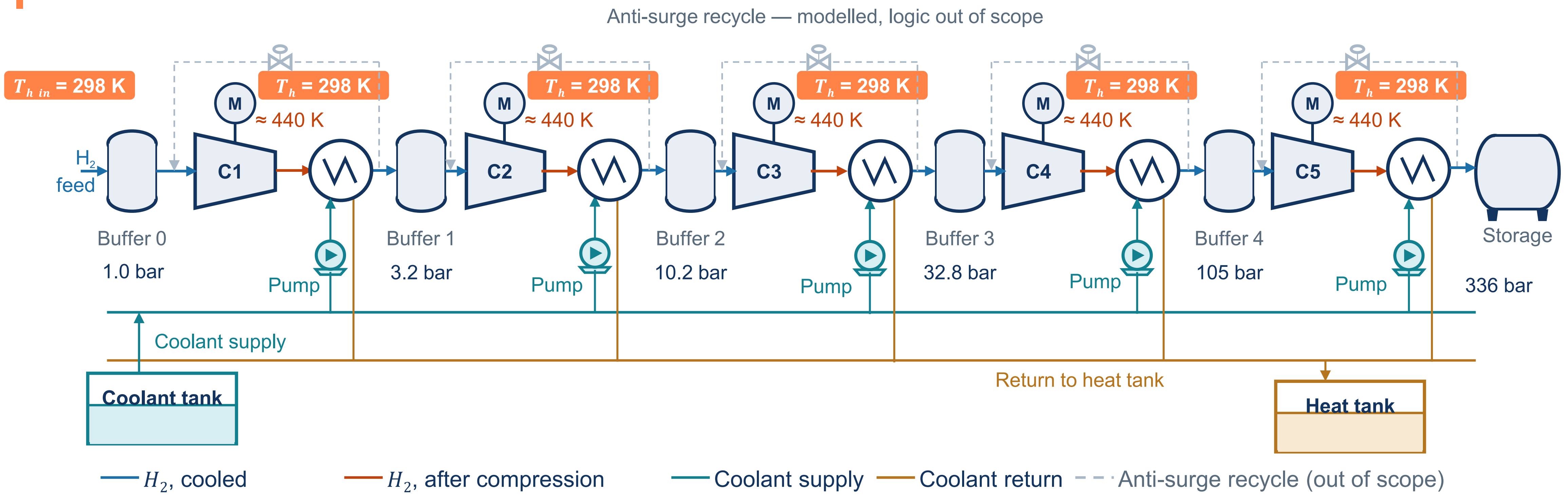


Density: why we compress, and why we cool between the stages



Normal hydrogen, Leachman equation of state (CoolProp); Aziz, Energies 14 (2021) 5917

The plant: a simplified hydrogen compression–intercooling system



Total number of stages is found by: $N = \left\lceil \frac{\ln(PR_{total})}{\ln(PR_{stage})} \right\rceil$

Five centrifugal stages in series, PR = 3.2

Why a robust design is needed: four properties of the plant

1

Uncertain, time-varying supply

Electrolyser output follows the weather. In this study the hydrogen flow steps 2.65 → 2.75 → 2.45 → 2.65 kg/s during the run.

2

Biased and noisy sensors

Inlet temperature measurements carry up to ±2 %, the flow measurement ±4 %. Critically, these do not shrink as the regulation error shrinks.

3

Nonlinear and coupled

The input multiplies the state, the heat-transfer coefficient drifts, and each stage's outlet is the next stage's inlet.

4

One soft actuator per stage

A coolant pump. Its command has to stay bounded and smooth — a pump is not a switch.

CONSEQUENCE

Uncertainty that does not vanish at the setpoint means exact asymptotic regulation is simply not available.
The question is: how small a residual error can we guarantee?

Existing work and research gap

Compressor control — map-based operability

- **Bøhagen & Gravdahl (2008)** active surge control via drive torque
- **Cortinovis et al. (2012)** MPC anti-surge, variable-speed drives
- **Bentaleb et al. (2015)** MPC for pressure regulation and surge
- **Milosavljevic et al. (2020)** real-time load sharing under uncertainty
- **Alsuwian et al. (2023)** review of anti-surge control systems

Shared assumption: the cooler outlet is a boundary condition

Heat-exchanger control — one thermal unit

- **Alvarez-Ramirez et al. (1997)** robust control on an extended-state observer
- **Zavala-Río et al. (2009)** bounded positive control, double-pipe
- **Padhee et al. (2011)** IMC-PID on a shell-and-tube exchanger
- **Vasičkaninová & Bakošová (2016)** $H_2, H\infty$, mixed and μ-synthesis
- **Dyrska et al. (2023)** MPC with constraint removal

Shared assumption: the compressor making the hot stream is not modelled

THE GAP

Missing: a unified feedback treatment of the coupled compressor–exchanger train with explicit hot-stream temperature regulation. Train-level dynamic models do exist — **Modekurti et al. (2017), Dominic & Maier (2014)** — but their control question stays operability, and the cooler outlets are held as fixed boundary conditions.

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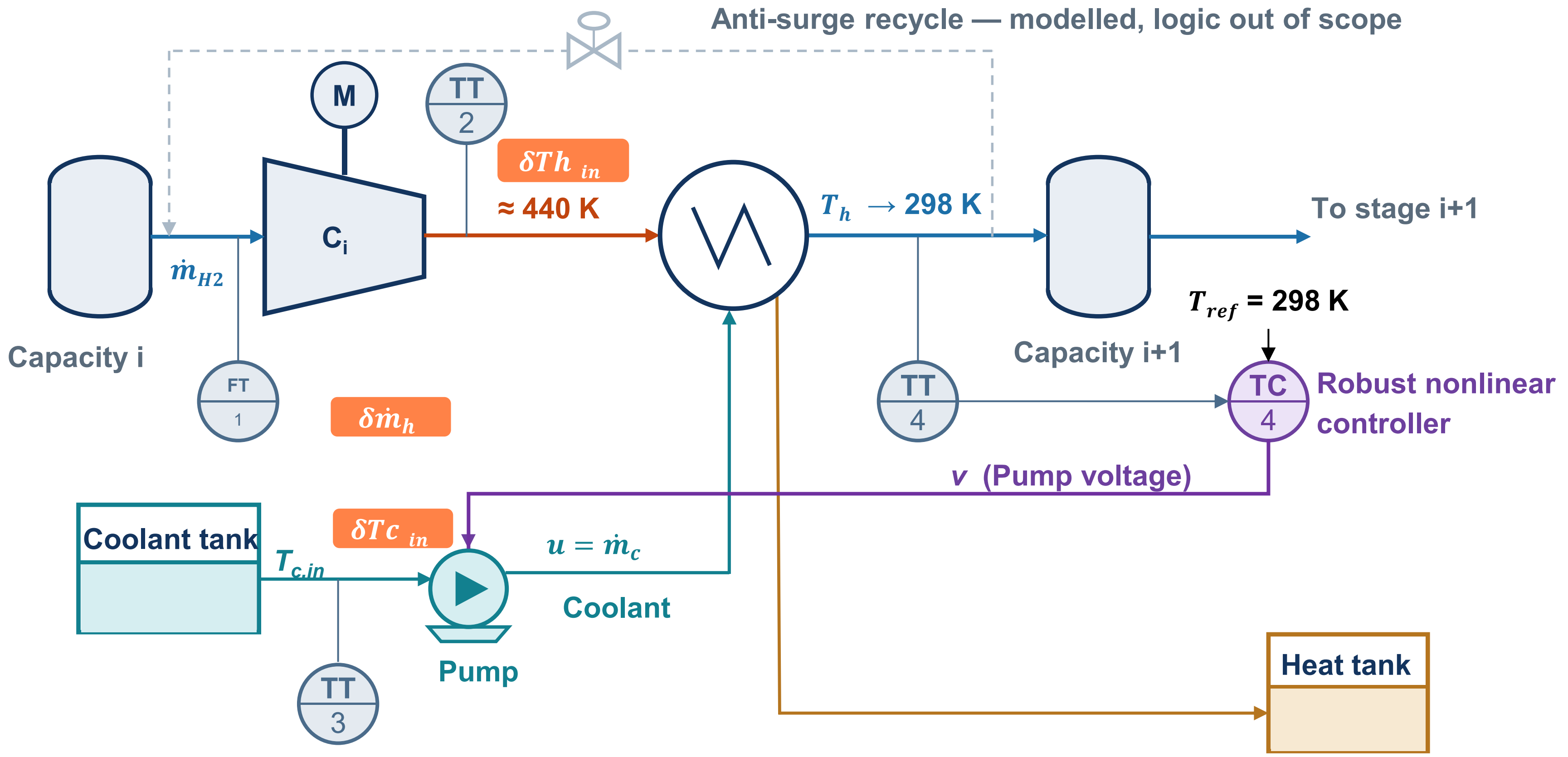
System description and control objective



Manipulated

Coolant mass flow $\dot{m}_c$, set by the pump voltage v

Controlled

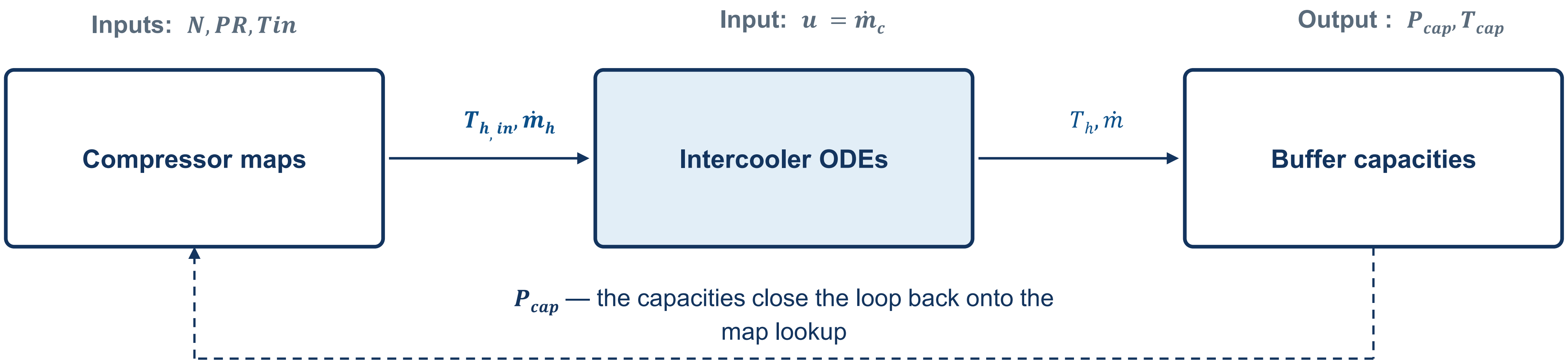
hot-side outlet T_h, held at 298 K

Uncertain

the three balloons marked δ — bounded, non-vanishing measurement error

Process line · Dashed = instrument signal · δ = Uncertain measurement · Balloons are ISA tags

Four sub-models, and the loop that ties them together



1 — Compressor map

Corrected flow, β

2 — Efficiency + coupling

$\eta_{is} \rightarrow T_{h,in}$

3 — Intercooler

Two lumped states

4 — Capacities

Mass + energy conservation model

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Control-oriented model: three coupled blocks

① COMPRESSOR

Quasi-static performance maps

$$\dot{m}_c = \dot{m}\frac{\sqrt{T_{in}/T_r}}{P_{in}/P_{ref}} \qquad N_c = \frac{N}{\sqrt{T_{in}/T_r}}$$

Corrected flow and speed → normalised pressure ratio → mass-flow map → isentropic-efficiency map.

A regularisation parameter β makes the map bijective where the speed lines flatten: $\beta = 1$ at surge, $\beta = 0$ at choke.

Discharge temperature feeds the intercooler hot side.

② INTERCOOLER

Lumped dynamic model — two states

$$\frac{d(T_h + T_{h,\mathrm{in}})}{dt} = \frac{2\dot{m}_h}{M_h}(T_{h,\mathrm{in}} - T_h) - \frac{UA_{hx}}{M_h C_{ph}}\,\Delta T_h$$

$$\frac{d(T_c + T_{c,\mathrm{in}})}{dt} = \frac{2\dot{m}_c}{M_c}(T_{c,\mathrm{in}} - T_c) + \frac{UA_{hx}}{M_c C_{pc}}\,\Delta T_h$$

$$\Delta T_h = T_{h,\mathrm{in}} + T_h - T_{c,\mathrm{in}} - T_c$$

$$T_{h,\mathrm{in}}(t) = T^n_{h,\mathrm{in}} + \delta T_{h,in}(t) \quad T_{c,\mathrm{in}}(t) = T^n_{c,in} + \delta T_{c,in}(t)$$

$$\dot{m}_h(t) = \dot{m}^n_h + \delta\dot{m}_h(t)$$

③ INTERSTAGE CAPACITY

Mass and energy balance

$$V\dot{\rho} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

$$M\dot{u}_s = \sum \dot{m}_{in} h_{in} - \sum \dot{m}_{out} h_{out} - \dot{m}_{net} u_s$$

Buffers damp mass and energy imbalances and let pressure and temperature evolve through accumulation instead of jumping.

They also recover system-level dynamics from a quasi-static compressor.

TIME-SCALE SEPARATION

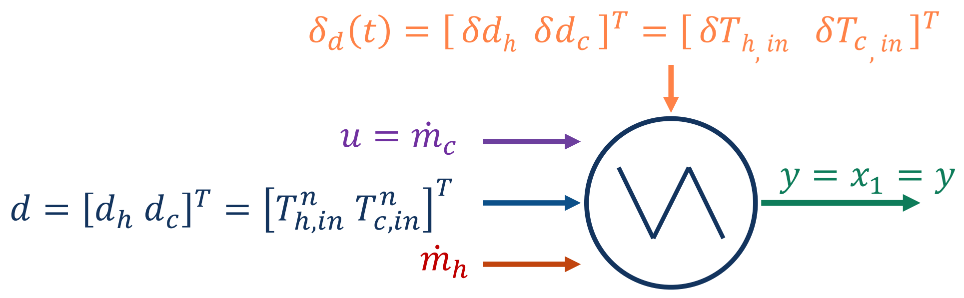
The regulated transient is governed by heat storage in the intercooler, not by rotor inertia or fluid momentum.

SISO state-space model of the intercooler and its uncertainty

② INTERCOOLER

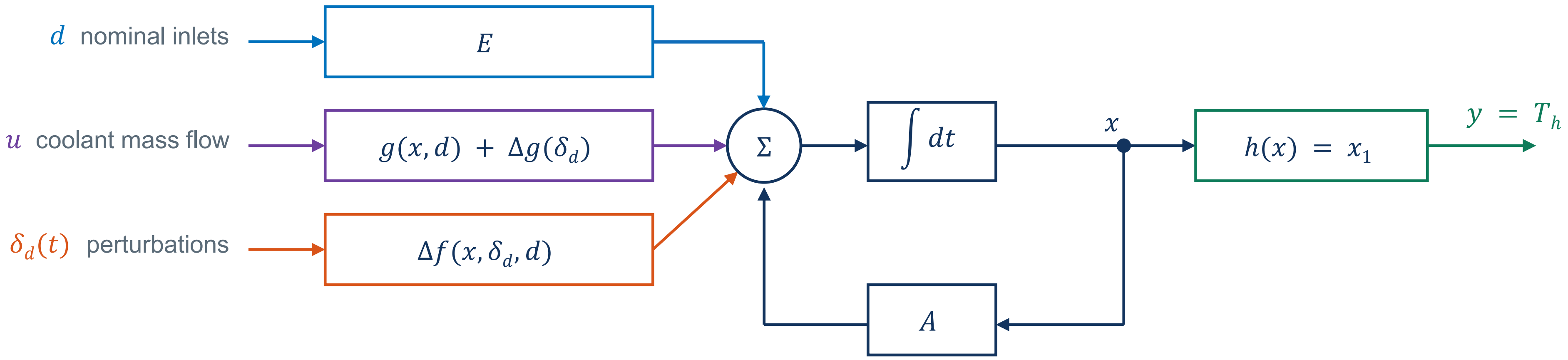
Lumped dynamic model — two states

$$\dot{x} = Ax + Ed + \Delta f(x, \delta_d, d) + [\, g(x,d) + \Delta g(\delta_d)]u,$$
$$y = h(x) = x_1$$



$$x = [\,x_1\;\; x_2\,]^T = [\,T_h\;\; T_c\,]^T$$

BLOCK DIAGRAM — THE MODEL AS IT IS SIMULATED



Obstacle to asymptotic stability: unmatched uncertainties

DECOMPOSITION THROUGH THE PROJECTOR

total uncertainty

Δf

$P_m = g\, g^+$ → $g\, \Delta f_m$

onto range g(x, d) — matched — cancelled exactly

$I - P_m$ → Δf_u

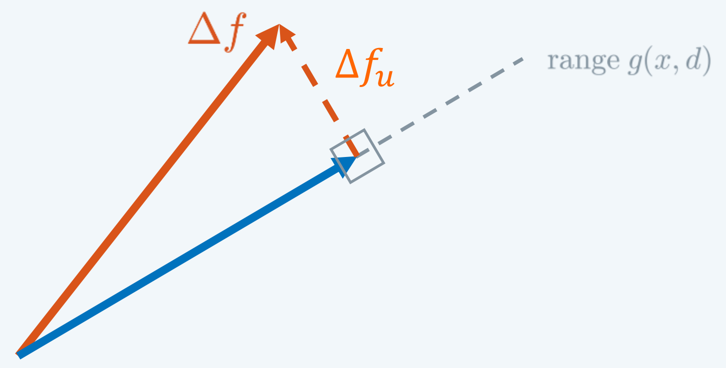
onto the complement — unmatched — only dominated

range g(x, d) is the cold-side row alone — nothing the pump does reaches the hot-side entry.

THE PROJECTION IS ORTHOGONAL — THE NORM SPLITS WITH IT

$$\|\Delta f\|^2 = \|g\, \Delta f_m\|^2 + \|\Delta f_u\|^2$$

PROJECTION DECOMPOSITION $P_m = g\, g^+$



$g\,\Delta f_m$ matched — cancelled exactly

Δf_u unmatched — only dominated

$$\Delta f = g\,\Delta f_m + \Delta f_u, \qquad \Delta g = g\,\Delta g_m$$

Controller design: nominal controller by input–output feedback linearization

Relative degree r = 2 = n

NOMINAL CONTROL LAW

$$u_n = \frac{-L_f^2 h(x) \;-\; k_1 h(x) \;-\; k_2 L_f h(x) \;+\; k_1 y_{\text{ref}}}{L_g L_f h(x)}$$

Full derivation — backup slides B1–B2

WELL-POSEDNESS

$$L_g L_f h(x) = -\frac{2a}{M_c}(x_2 - d_c) \;\neq\; 0$$

x_2 stays above the coolant supply temperature whenever the exchanger is actually transferring heat, so the decoupling term never loses rank in the operating region.

WHAT THE NOMINAL STEP BUYS

— Exponential decay of the regulation error at a guaranteed rate λ — for the nominal plant.

— A quadratic Lyapunov function from the Lyapunov equation, which the robust step then reuses unchanged.

— Gains here: $K = [0.02 \;\; 0.3]$, with $Q = I$.

Practical stability: the notions used

The stability result is stated in terms of practical stability, for the general nonlinear system

$$\frac{d\xi}{dt} = F(\xi, t), \qquad \xi(0) = \xi_0, \qquad t \in \mathbb{R}_{\geq 0}, \qquad \xi \in \mathbb{R}^n$$

DEFINITION 1 — LOCAL PRACTICAL ASYMPTOTIC STABILITY (LPAS) *Benabdallah, Ellouze & Hammami (2009)*

A solution of the system above is locally practically asymptotically stable if there exist a function β of class $\mathcal{KL}$, a constant $r > 0$ and a neighbourhood Ω_0 of the origin such that, for every initial condition ξ_0 in Ω_0,

$$\|\xi(t)\| \leq \beta(\|\xi_0\|, t) + r, \qquad \forall\, t \geq 0$$

DEFINITION 2 — LOCAL PRACTICAL EXPONENTIAL STABILITY (LPES) *Benabdallah, Ellouze & Hammami (2009)*

A solution of the system above is locally practically exponentially stable if there exist constants $\gamma > 0, \nu > 0, r > 0$and a neighbourhood Ω_0 of the origin such that, for every initial condition ξ_0 in Ω_0,

$$\|\xi(t)\| \leq \gamma\, \|\xi_0\|\, e^{-\nu t} + r, \qquad \forall\, t \geq 0$$

The constant r is called the ultimate bound.

Main result: practical exponential stability

ASSUMPTION 2 — UNCERTAINTY SET

The time-varying uncertainty vector is continuous and its values lie in a compact set:

$$\delta_u(t) \in D \subset \mathbb{R}^2 \quad \forall\, t \geq 0, \qquad D \ \text{compact}$$

No statistics are assumed — only a bound.

ASSUMPTION 3 — KNOWN BOUNDING FUNCTIONS

There exist known continuous, locally Lipschitz bounding functions such that, for all x in Ω,

$$\|\Delta f_m\| \leq \rho_m, \qquad \|\Delta f_u\| \leq \rho_u, \qquad \|\Delta g_m\| \leq \rho_{gm} < 1$$

The third bound, strictly below one, keeps the input gain positive.

THEOREM 1 — LOCAL PRACTICAL EXPONENTIAL STABILITY

Consider the regulation error $\varepsilon = x - x_{\mathrm{ss}}$ and its closed-loop dynamics

$$\dot{\varepsilon} = f_n(\varepsilon, d) + g(u_r + \Delta g_m u_n + \Delta g_m u_r + \Delta f_m) + \Delta f_u$$

and let P_ε be a symmetric positive-definite matrix defining the quadratic Lyapunov function V. Suppose Assumptions 2 and 3 hold in a domain Ω_0 contained in Ω. Then, under the control law $u = u_{\mathrm{n}} + u_{\mathrm{r}}$, with the nominal law of the preceding slide and the robust compensator defined on the next one, the closed-loop error system is locally practically exponentially stable in the sense of Definition 2. Moreover, for all $t \geq 0$,

$$\|\varepsilon(t)\| \;\leq\; \sqrt{\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)}}\; \|\varepsilon(0)\|\, e^{-\frac{\lambda}{2}t} \;+\; 2\sqrt{\frac{\kappa}{\lambda_{\min}(P_\varepsilon)(\varrho - \lambda)}} \;+\; \sqrt{\frac{2}{\delta\lambda\lambda_{\min}(P_\varepsilon)}}\; sup_{s\in[0,t]}\sqrt{\rho_{gm}(s)}$$

Proof — backup slides B5–B7

Controller design: robust controller with practical stability guarantees

$$u_r = u_{r1} + u_{r2} - \delta \|u_n\|^2 g^T P_\varepsilon \varepsilon$$

$$u_{r1} = -\frac{\rho_m \, \mu_1}{\eta(\|\mu_1\| + \kappa e^{-\varrho t})}$$

cancels the MATCHED part

$g\,\Delta f_m$, using the bound ρ_m

$$u_{r2} = -\frac{\rho_u \|P_\varepsilon \varepsilon\|}{\eta \|g^T P_\varepsilon \varepsilon\|} \cdot \frac{\mu_2}{\|\mu_2\| + \kappa e^{-\varrho t}}$$

attenuates the UNMATCHED part

Δf_u, using the bound ρ_u

$$-\,\delta\,\|u_n\|^2\, g^T P_\varepsilon \varepsilon$$

dominates the UNCERTAIN INPUT GAIN

Δg_m, via Young's inequality

BENCHMARK — HEJRI (2025): SWITCHED

The normalisation $\|\mu\| \to 0$ as $\varepsilon \to 0$, so u_r must be capped by a switching rule. The result is a piecewise-defined feedback law — and chattering in the pump command.

THIS WORK — CONTINUOUS (AFTER QU, 1992)

The same singularity is regularised by a strictly positive vanishing term $\kappa e^{-\varrho t} > 0$ in the denominator. u_r stays bounded and continuous everywhere — there is no switching surface at all.

The benchmark controller: Hejri (2025)

$$u_r = -\frac{N}{D}\, g^T \nabla V \quad (D > \varepsilon), \qquad u_r = -\frac{N}{\varepsilon}\, g^T \nabla V \quad (D \le \varepsilon)$$

THE NUMERATOR — BOTH FAMILIES OF BOUNDS

$$N = \|\nabla V\|(\rho_u + \rho_{gu}\|u_n\|)$$

$$+ \ \|g^T \nabla V\|(\rho_m + \rho_{gm}\|u_n\|)$$

Matched and unmatched bounds sit in one gain — this is what separates it from a matched-only design.

THE DENOMINATOR — WHERE IT GOES SINGULAR

$$D = \|g^T \nabla V\| \ \left(\|g^T \nabla V\|(1 - \rho_{gm}) - \|\nabla V\|\rho_{gu}\right)$$

D vanishes on the null set, so the gain cannot simply be N/D: it has to be capped, and ε is where the cap takes over.

AS IMPLEMENTED ON THE INTERCOOLER

The coolant flow carries no input-gain uncertainty, $\rho_{gu} = 0$, so

$$D = s = \|g^T \nabla V\|^2(1 - \rho_{gm})$$

with the same bounds and the same nominal law as ours.

WHERE THE CHATTERING COMES FROM

The two branches meet only in the limit, so the gain jumps as the state crosses s = ε — and the pump command inherits the jump. Only the gain structure differs from ours; everything upstream of it is identical.

Reading the bound: the ultimate bound r is a design parameter

THE BOUND OF THEOREM 1, IN THE FORM OF DEFINITION 2

$$\|\varepsilon(t)\| \leq \gamma \|\varepsilon(0)\| e^{-\nu t} + r$$

$$\gamma = \sqrt{\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)}}, \qquad \nu = \frac{\lambda}{2}$$

WHERE THE ULTIMATE BOUND r COMES FROM

$$\sqrt{\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)}} \|\varepsilon(0)\| e^{-\frac{\lambda}{2}t}$$

TRANSIENT

+

$$2\sqrt{\frac{\kappa}{\lambda_{\min}(P_\varepsilon)(\varrho - \lambda)}}$$

REGULARISATION RESIDUE

+

$$\sqrt{\frac{2}{\delta\,\lambda\,\lambda_{\min}(P_\varepsilon)}} \sup_{s\in[0,t]} \sqrt{\rho_{gm}(s)}$$

INPUT-UNCERTAINTY RESIDUE

Proof — backup slides B5–B7

Simulation setup: plant, uncertainty and tuning

PLANT

Stages · pressure ratio each	$5 \cdot 3.2$
Inlet → discharge pressure	$1\ bar \rightarrow \approx 335\ bar$
Overall conductance UA_{hx}	$80 \times 10^3\ W/K$
Hold-up masses M_h, Mc	$100\ kg\ each$
$C_{ph}\ (H_2) \cdot C_{pc}$ (water)	$14\,492 \cdot 4\,186\ J/kg \cdot K$
Nominal $T_{h,in} \cdot T_{c,in}$	$298\ K \cdot 273.15\ K$
Initial state · setpoint	$[320, 320]\ K \cdot 298\ K$

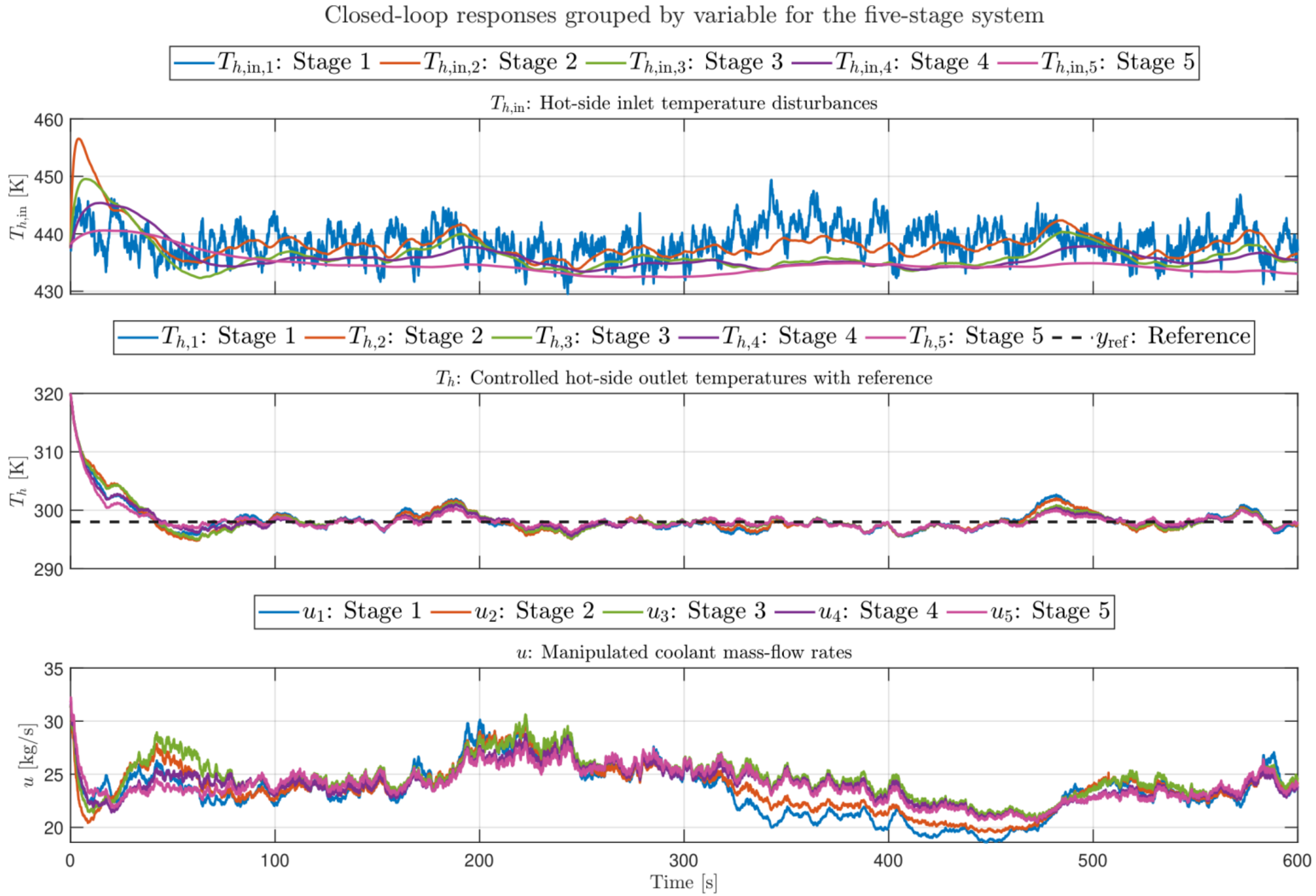
UNCERTAINTY AND TUNING

$\delta d_h, \delta d_c$ (multiplicative)	$U[-0.02, 0.02]$
$\delta \dot{m}_h$	$U[-0.04, 0.04]$
Feedback gain K · weight Q	$[0.02\ \ 0.3] \cdot I$
Regularisation κ	10^5
Young parameter δ	10^{-6}
Decay offset ϱ	$1.1\ \lambda$
Setpoint step at $t = 300\ s$	$298\ K \rightarrow 288\ K$

SCENARIO — HYDROGEN MASS FLOW AND SETPOINT

0 s	150 s	300 s	450 s	600 s
2.65 kg/s	2.75 kg/s	2.45 kg/s + setpoint 288 K	2.65 kg/s	

Closed-loop results: temperature regulation



TOP — the disturbance

Hot-side inlet temperature at each stage, 430–460 K and moving. This is what the controller has to reject.

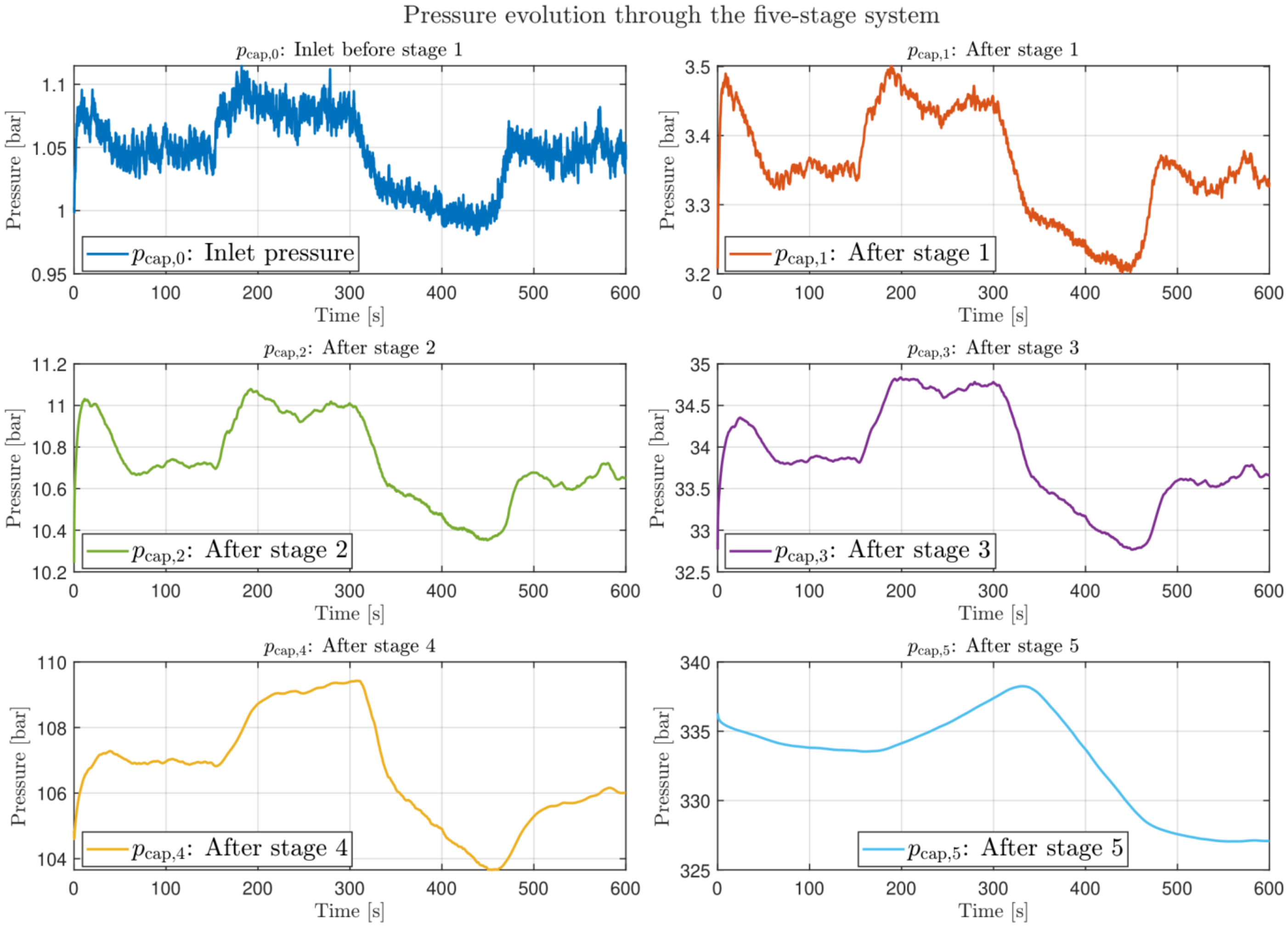
MIDDLE — the result

Every stage's outlet temperature converges into a tight band around y_{ref} and stays there across all three flow steps.

BOTTOM — the cost

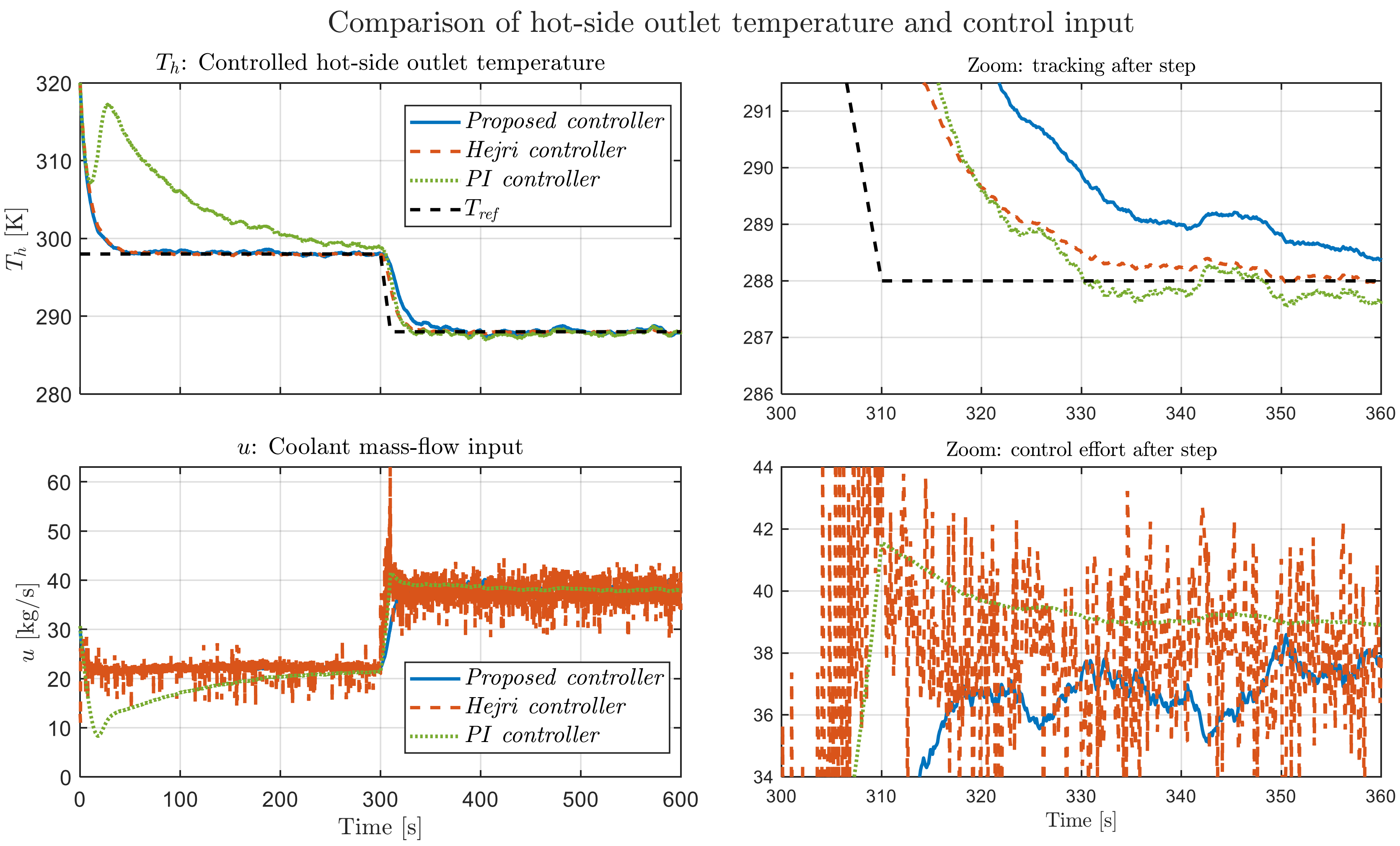
Coolant flows stay bounded between roughly 20 and 35 kg/s, with no chattering and no saturation.

Closed-loop results: pressure consistency



- Stage pressures follow the design ladder: **1 → 3.3 → 10.7 → 34 → 108 → 335 bar**.
- Transients are smooth.
- The temperature loop does not disturb the compression profile.

Comparison of controllers: hot-gas temperature



TRACKING

- **PI** overshoots to 320 K and needs ≈ 200 s to recover.
- ***Hejri*** and **proposed** controllers both track closely, before and after the setpoint step.

CONTROL EFFORT

- ***Hejri*** controller spikes to 65 kg/s with sustained high-frequency switching.
- **Proposed** controller is smooth at a comparable mean — the spikes are simply gone.

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Backup slides

Derivations and proofs, in full — shown on request during the discussion.

B1 · Nominal design (1/2): input–output feedback linearization

(1) Differentiating the output along the vector field f, we obtain

$$\dot{y} = L_f h(x)$$

(2) The control input does not yet appear, so we differentiate once more, which yields

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x)\, u_n$$

(3) Expanding the second Lie derivative in the state and the measured inlet conditions gives

$$L_f^2 h(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 d_h + \alpha_4 d_c$$

(4) The decoupling term never loses rank in the operating region, since the coolant outlet stays below the hot-side state whenever the exchanger is transferring heat,

$$L_g L_f h(x) = -\frac{2a}{M_c}(x_2 - d_c) \neq 0$$

(5) so that the relative degree equals the system order and no internal dynamics remain:

$$r = 2 = n \;\Rightarrow\; \text{no zero dynamics}$$

B2 · Nominal design (2/2): error coordinates and gain placement

(1) In the linearizing coordinates the dynamics form a chain of integrators,

$$z = [\ h(x) \quad L_f h(x)\ ]^T, \qquad \dot{z} = A_c z + B_c v$$

(2) and choosing the outer loop as

$$v = -k_1 z_1 - k_2 z_2 + k_1 y_{\text{ref}}$$

(3) renders the tracking error linear with a Hurwitz characteristic polynomial:

$$\dot{e}_z = (A_c - B_c K)\, e_z, \qquad s^2 + k_2 s + k_1 \ \text{ Hurwitz}$$

(4) Solving for the input gives the nominal control law

$$u_n = \frac{-L_f^2 h(x) \ - \ k_1 h(x) \ - \ k_2 L_f h(x) \ + \ k_1 y_{\text{ref}}}{L_g L_f h(x)}$$

(5) With P taken from the Lyapunov equation, the nominal part of the closed loop decays at a guaranteed rate:

$$\varepsilon^T P_\varepsilon f_n \le -\lambda V, \qquad \lambda = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0$$

B3 · Uncertainty decomposition: what the coolant flow can and cannot reach

(1) The pseudo-inverse of the input vector field defines the projection onto its range,

$$g^{+} = (g^{T} g)^{-1} g^{T}, \qquad P_m = g\, g^{+}$$

(2) so that every uncertainty splits into a matched and an unmatched part:

$$\Delta f = g\, \Delta f_m + \Delta f_u, \qquad \Delta g = g\, \Delta g_m$$

(3) The matched and unmatched parts are given,

$$\Delta f_m = -\frac{UA_{hx}}{2C_{pc}} \frac{\delta d_h - \delta d_c}{x_2 - d_c}, \qquad \Delta f_u = \begin{bmatrix} \frac{2\dot{m}_h^n}{M_h}\delta d_h + \frac{2\delta\dot{m}_h}{M_h}(d_h + \delta d_h - x_1) - a(\delta d_h - \delta d_c) \\ 0 \end{bmatrix}$$

(4) and the input-gain uncertainty is matched by construction:

$$\Delta g_m = -\frac{\delta d_c}{x_2 - d_c} \qquad \Delta g_u = 0$$

(5) All three parts are bounded, the input-gain bound being strictly less than one,

$$\|\Delta f_m\| \le \rho_m, \qquad \|\Delta f_u\| \le \rho_u, \qquad \|\Delta g_m\| \le \rho_{gm} < 1$$

(6) which makes the effective input gain uniformly positive:

$$0 < \eta \le \inf_{(\varepsilon, t)}(1 + \Delta g_m)$$

B4 · The robust controller and its three terms

(1) The applied input is the nominal law plus a robust correction,

$$u = u_n + u_r$$

(2) where the robust part is built from three terms:

$$u_r = u_{r1} + u_{r2} - \delta\|u_n\|^2 g^T P_\varepsilon \varepsilon$$

(3) The first cancels the matched uncertainty,

$$u_{r1} = -\frac{\rho_m\,\mu_1}{\eta(\|\mu_1\| + \kappa e^{-\varrho t})} \quad , \quad \mu_1 = \rho_m g^T P_\varepsilon \varepsilon$$

(4) the second attenuates the unmatched uncertainty,

$$u_{r2} = -\frac{\rho_u\|P_\varepsilon \varepsilon\|}{\eta\|g^T P_\varepsilon \varepsilon\|} \cdot \frac{\mu_2}{\|\mu_2\| + \kappa e^{-\varrho t}} \quad , \quad \mu_2 = \frac{\rho_u\|P_\varepsilon \varepsilon\|}{\|g^T P_\varepsilon \varepsilon\|}\, g^T P_\varepsilon \varepsilon$$

(5) and the third dominates the uncertain input gain:

$$-\,\delta\|u_n\|^2\, g^T P_\varepsilon \varepsilon$$

(6) The regularisation stays strictly positive for all time, so no switching surface is ever reached:

$$\kappa e^{-\varrho t} > 0 \quad \forall t \geq 0$$

B5 · Proof (1/3): the Lyapunov derivative

(1) Consider the quadratic Lyapunov function

$$V = \frac{1}{2}\varepsilon^T P_\varepsilon \varepsilon, \qquad P_\varepsilon = P_\varepsilon^T > 0$$

(2) Along the perturbed error dynamics

$$\dot{\varepsilon} = f_n(\varepsilon, d) + g(u_r + \Delta g_m u_n + \Delta g_m u_r + \Delta f_m) + \Delta f_u$$

(3) its derivative separates into the nominal, unmatched and matched contributions:

$$\dot{V} = \varepsilon^T P_\varepsilon f_n + \varepsilon^T P_\varepsilon \Delta f_u + \varepsilon^T P_\varepsilon g[\Delta g_m u_n + \Delta f_m + (1 + \Delta g_m) u_r]$$

(4) The nominal term supplies the exponential rate,

$$\varepsilon^T P_\varepsilon f_n \leq -\lambda V, \qquad \lambda = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0$$

(5) and each regularised robust term is bounded by the elementary inequality

$$\frac{\|\mu_i\| \, \kappa e^{-\varrho t}}{\|\mu_i\| + \kappa e^{-\varrho t}} \leq \kappa e^{-\varrho t}$$

B6 · Proof (2/3): Young's inequality and the differential inequality

(1) Young's inequality with parameter δ states that

$$\psi w \le \frac{1}{2\delta}\psi^2 + \frac{\delta}{2}w^2 \qquad 0 \le \frac{\psi w}{\psi + w} \le w$$

(2) Applied to the cross term produced by the uncertain input gain, it gives

$$(1+\Delta g_m)\|u_n\|\|g^T P_\varepsilon \varepsilon\| \le \frac{\rho_{gm}}{\delta} + \delta(1+\Delta g_m)\|u_n\|^2\|g^T P_\varepsilon \varepsilon\|^2$$

(3) The third robust term was chosen precisely to cancel the remaining quadratic, which leaves the differential inequality

$$\dot{V} \;\le\; -\lambda V \;+\; 2\kappa e^{-\varrho t} \;+\; \frac{\rho_{gm}}{\delta}$$

(4) valid for every trajectory in the operating region, with the bounds

$$\|\Delta f_m\| \le \rho_m, \qquad \|\Delta f_u\| \le \rho_u, \qquad \|\Delta g_m\| \le \rho_{gm} < 1$$

B7 · Proof (3/3): integration and the ultimate bound

(1) Applying the comparison lemma to the differential inequality yields

$$V(t) \le e^{-\lambda t} V(0) + \frac{2\kappa}{\varrho - \lambda} + \frac{1}{\lambda\delta} \sup_{s\in[0,t]} \rho_{gm}(s)$$

(2) Converting back from V to the error norm gives the three-part bound

$$\|\varepsilon(t)\| \;\le\; \sqrt{\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)}}\;\|\varepsilon(0)\|\, e^{-\frac{\lambda}{2}t} \;+\; 2\sqrt{\frac{\kappa}{\lambda_{\min}(P_\varepsilon)(\varrho - \lambda)}} \;+\; \sqrt{\frac{2}{\delta\lambda\lambda_{\min}(P_\varepsilon)}}\; \sup_{s\in[0,t]} \sqrt{\rho_{gm}(s)}$$

(3) which is exactly local practical exponential stability in the sense of Definition 2,

$$\|\varepsilon(t)\| \;\le\; \gamma\, \|\varepsilon(0)\|\, e^{-\nu t} \;+\; r$$

(4) with the transient constant and the decay rate given by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)}}, \qquad \nu = \frac{\lambda}{2}$$

(5) and the residual radius r is the sum of the two residues — it shrinks as κ is reduced and δ is raised, so r is a design dial rather than a fixed penalty:

$$r = 2\sqrt{\frac{\kappa}{\lambda_{\min}(P_\varepsilon)(\varrho - \lambda)}} \;+\; \sqrt{\frac{2}{\delta\,\lambda\,\lambda_{\min}(P_\varepsilon)}}\; \sup_{s\in[0,t]} \sqrt{\rho_{gm}(s)}$$