# A Parametric Sensitivity Based Arrival Cost State Estimator


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## 1. A Priori Propagation

By setting $N=0$, the MHE problem is only solved for the current time.

To include a linearization point for the next $\Gamma^A$, we use an a priori expectation of the states next value:

$$x_{k+1}^{-}=F(x_k^{+})$$

## Indices

The indexes are written in discrete time, using a fixed reference frame.

This differs from the Background, as each optimization problem can have its own reference frame.

## Background

This work is based on the Moving Horizon Estimator (MHE), which estimates the process noise, w; measurement noise, $v$; and states, $x$, in a time-window with the current time, $T$, and the $N$ previous time instances as an optimization problem*:

$$\min_{x,v,w} \sum_{k=T-N}^{T-1} w_k^T W w_k + \sum_{k=T-N}^{T} v_k^T V v_k + \sum_{k=T-N}^{T} \zeta_k^T Z \zeta_k + \Gamma_{T-N}^{A}(\hat{x}_{T-N})$$

$$\begin{aligned} s.t. \quad & y_k = H x_k + v_k, && k=T-N \dots T \\ & x_{k+1} = F(x_k) + w_k, && k=T-N \dots T-1 \\ & e_k = J_k x_k + \zeta_k, && k=T-N \dots T \end{aligned}$$

The Arrival Cost, $\Gamma$, permits the use of an $N$-sized horizon, instead of the whole history, as they can be decoupled by using the separation principle.
This decoupling can be repeated, which allows us to formulate the optimization problem for $\Gamma$ as:

$$\Gamma_k(x_k) = \min_{x,v,w} \sum_{i=k-M}^{k-1} w_i^T W w_i + \sum_{i=k-M}^{k-1} v_i^T V v_i + \sum_{i=k-M}^{k-1} \zeta_i^T Z \zeta_i + \Gamma_{k-M}(x_{k-M})$$

$$\begin{aligned} s.t. \quad & y_i = H x_i + v_i, && i=k-M \dots k-1 \\ & x_{i+1} = F(x_i) + w_i, && i=k-M \dots k-1 \\ & e_i = J_i x_i + \zeta_i, && i=k-M \dots k-1 \end{aligned}$$

To avoid nesting optimization problems, the parametric nonlinear sensitivity approach of Bjorvand (2025) is used, where we approximate $\Gamma$ as a quadratic function:

$$\Gamma_k^A(x_k) = (x_{k-M} - \tilde{x}_{k-M})^T X_{k-M} (x_{k-M} - \tilde{x}_{k-M})$$

The aim is to use this sensitivity-based approach and simplify it to a version that can be solved explicitly.

*Here, a linearization of the constraints is added, $e_i = J_i x_i$, where $\zeta$ penalizes deviating from the linearization point.

These optimization problems can have additional equality and inequality constraints.

## 2. Arrival Cost Problem

By setting $M=1$, the Approximate Arrival Cost problem can be included in the MHE problem.

As such we can identify the following solution:
$x_k^{opt}=x_k^{-}$, $v_k^{opt}=v_k$, $w_k^{opt}=0$, $\zeta^{opt}=0$.

## KKT Conditions

for an arrival cost problem is:

$$\nabla_\omega L(x_{k+1}, \xi^{opt}) = 0,$$
$$c(x_{k+1}, \omega^{opt}) = 0,$$
$$-g_i(x_{k+1}, \omega^{opt}) = 0 \vee \mu_i^{opt} = 0,\ i = 1 \dots |g|,$$

where $c$ are the equality constraints, $g$ are the inequality constraints, $\lambda$ and $\mu$ are parameters, and

$$L = \Gamma_k(x_k) + c^T \lambda + g^T \mu,$$
$$\omega = (w_k, v_k, x_k),$$
$$\xi = (\omega, \lambda, \mu).$$

## 3. Parametric Sensitivities

We can find the local sensitivities through the KKT conditions:

$$S_{x_k} = \frac{\partial x_k^{opt}}{\partial x_{k+1}^{-}},\ S_{v_k} = \frac{\partial v_k^{opt}}{\partial x_{k+1}^{-}},$$
$$S_{w_k} = \frac{\partial w_k^{opt}}{\partial x_{k+1}^{-}},\ S_{\zeta_k} = \frac{\partial \zeta_k^{opt}}{\partial x_{k+1}^{-}},$$

whom all depend on the active constraints.

## Parametric Sensitivities

can be evaluated via the implicit function theorem, which states that if

$$F(x,y)=0,$$

then the local derivative is

$$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial y}\right)^{-1} \times \left(\frac{\partial F}{\partial x}\right).$$

For the arrival cost problem, this is:

$$F = \begin{bmatrix} \nabla_\omega L(x_{k+1}, \xi^{opt}) \\ c(x_{k+1}, \omega^{opt}) \\ -g_+(x_{k+1}, \omega^{opt}) \end{bmatrix} = 0$$

where $g_+$ is the active constraints.

## Assumptions

- The measurement function is linear, or can be linearized.
- The constraints are linear, or can be linearized
- The state transition function is first order differentiable.
- The true solution lies in the neighborhood of the a priori solution.
- Any active inequality constraints are strongly active.

## Constraints

Assuming the true solution is close to the a priori, then we linearize the constraints:

$$e_k = J_k x_k^{-},\ J_k = \left.\frac{do}{dx}\right|_{\omega=(w_k, v_k, x_k^{-})}$$

where, $o=(c,g)$.

And $\zeta$ decides the deviation.

## 4. Arrival Cost Update

With the sensitivities, the current $\Gamma$ can be linearized by Bjorvand's update rule:

$$X_{k+1} = S_{w_k}^T W S_{w_k} + S_{v_k}^T V S_{v_k} + S_{\zeta_k}^T X_k S_{\zeta_k} + S_{x_k}^T J_k S_{x_k}$$

$$\tilde{x}_{k+1} = \hat{x}_{k+1}^{-} - X_{k+1}^{-1} \begin{pmatrix} S_{w_k}^T W w_k^{opt} + S_{v_k}^T V v_k^{opt} + S_{\zeta_k}^T V v_\zeta^{opt} \\ + S_{x_k}^T X_k (w_k^{opt} - \tilde{x}_k) \end{pmatrix}$$

## 5. A Posteriori Estimate

The state estimate can be updated with the measurement by solving the MHE problem.

Assuming a linear measurement function and $N=0$, the closed form solution is:

$$x_{k+1}^{+} = (X_{k+1} + H_{k+1}^T V H_{k+1} + J_{k+1}^T Z_{k+1} J_{k+1})^{-1} \times (X_{k+1} \tilde{x}_{k+1} + H_{k+1}^T V_{k+1} y_{k+1} + J_{k+1}^T Z_{k+1} e_{k+1})$$

## Main Conclusion

- A novel explicit state estimation method has been derived.
- This method uses information about the constraints.
  - It will be biased toward the a priori solution the larger the derivative of the constraints are.
- It has given identical results to the Extended Kalman Filter in cases without constraints.

## Future Work

- Modify the first step to ensure that the a priori state will be feasible.
- Test this method on cases with active constraints.
- Check how this method relates to the Extended Kalman Filter
  - Check if it reduces to the Extended Kalman Filter without constraints.
- Investigate how to dynamically decide the weights in $Z$.

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**Acknowledgments:**

This work was carried out as a part of SUBPRO-Zero, a Research Centre at the Norwegian University of Science and Technology (project code 100589 and CristinID 2720670). The authors gratefully acknowledge the project support from SUBPRO-Zero, which is financed by major industry partners and NTNU.



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